Distributed Delays Facilitate Amplitude Death of Coupled Oscillators

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Coupled oscillators are shown to experience amplitude death for a much larger set of parameter values when they are connected with time delays distributed over an interval rather than concentrated at a point. Distributed delays enlarge and merge death islands in the parameter space. Furthermore, when the variance of the distribution is larger than a threshold the death region becomes unbounded and amplitude death can occur for any average value of delay. These phenomena are observed even with a small spread of delays, for different distribution functions, and an arbitrary number of oscillators. (http://link.aps.org/abstract/PRL/v91/e094101)

Coupled oscillators constitute an effective and popular paradigm for the study of interacting oscillatory processes in the physical and biological sciences [1]. The rich dynamics arising from the interaction of simple units have been a source of interest for scientists modeling the collective behavior of real-life systems. Among the most widely studied phenomena is synchronization, where individual units oscillate at a common frequency and phase when coupled [2]. Synchronization may be observed even under weak coupling; so it has usually been studied through reduced models that retain only the phase information along the limit cycles. With stronger coupling further interesting behavior is possible, whose investigation requires the use of full models that include the amplitudes of the oscillators. An example is amplitude death, which refers to the quenching of oscillations under coupling, as the system evolves to a stable equilibrium [3, 4]. If the information flow through the coupled system is instantaneous, amplitude death occurs when the individual oscillators have sufficiently different frequencies. [5, 6, 7]. On the other hand, if the information from one oscillator reaches the others after a certain time delay, which may be due to finite propagation or information processing speeds, then even identical oscillators can experience amplitude death when coupled [8]. Recent experimental and theoretical studies have confirmed the role of delays in inducing amplitude death [9, 10, 11].

While the importance of time delays in amplitude death is now clear, studies in this area have so far been confined only to discrete, or constant, delays. In other words, it has been assumed that information reaches from one unit to another after a fixed time τ which is unchanging as the system evolves, and moreover, the units act only on the instantaneous value of the received information and forget any previous values. Such discrete-delay models often fail to adequately describe physical systems by neglecting the possibilities that (a) the quantity τ may only be approximately known, (b) it may only represent an average value of a quantity that varies between pairs of oscillators in a network or (c) varies in time through a process involving unmodelled factors, and (d) the oscilla-

tors may incorporate "memory" effects by using the past history of the received information. The first possibility is certainly an issue in any experimental situation, (b) is typical when considering large arrays, and (c) and (d) are particularly significant in biology and neurology. Because of these shortcomings, models based on *distributed* delays have been proposed as early as the time of Volterra [12], and used in such areas as biology [13], ecology [14, 15], neurology [16], viscoelasticity [17], and economics [18]. It has especially been pointed out in the biological sciences that distributed delays lead to more realistic models [19]. In this Letter we consider the effects of distributed delays on amplitude death, and show that even a small spread in the delay distribution can greatly enlarge the set of parameters for which amplitude death occurs.

The oscillators studied here are described by

$$\dot{Z}(t) = (1 + i\omega_0 - |Z(t)|^2)Z(t), \qquad (1)$$

where $Z(t)$ is a complex number and $\omega_0 > 0$. Equation (1) represents the normal form for a supercritical Hopf bifurcation, and has been used to describe limit cycle oscillators where oscillations arise through such a bifurcation. It has an unstable equilibrium at zero, and an attracting limit cycle $Z(t) = \exp(i\omega_0 t)$ with frequency ω_0. Suppose a pair of such oscillators are coupled with general time delays:

$$\dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t)$$
$$+ K\left[\int_0^\infty f(\tau')Z_2(t - \tau')d\tau' - Z_1(t)\right] \quad (2)$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t)$$
$$+ K\left[\int_0^\infty f(\tau')Z_1(t - \tau')d\tau' - Z_2(t)\right]. \quad (3)$$

Here, K is a number quantifying the strength of coupling, and f represents a distribution of delay values. When f is the delta function $\delta(0)$ one obtains the system considered in [5], where the oscillators interact without delay. Similarly, the choice $\delta(\tau)$ with $\tau > 0$ gives the system with

a discrete delay which was studied in [8]. In the general case f is a probability density over an appropriate interval, which addresses the shortcomings mentioned in the above paragraph. We shall show that the variance of f has a significant effect on the dynamics of the system.

When the system (2)–(3) experiences amplitude death its zero solution becomes stable. To investigate the stability, the system is linearized about zero. The characteristic equation is found by making the ansatz $(Z_1(t), Z_2(t)) = \xi \exp(\lambda t)$, $\xi \in \mathbf{R}^2$, and is given by

$$(1 + i\omega_1 - K - \lambda)(1 + i\omega_2 - K - \lambda) - K^2 \left[F(\lambda)\right]^2 = 0 \quad (4)$$

where F is the Laplace transform of f. For definiteness, the analysis of (4) will be illustrated for uniformly distributed delays over the interval $\tau \pm \alpha$, i.e. $f(\tau') = 1/(2\alpha)$ if $|\tau - \tau'| \le \alpha$ and zero otherwise. However, the idea is the same for other types of distributions. We first consider oscillators with identical frequencies, $\omega_1 = \omega_2 = \omega_0$. This is the more stringent case for stability, because different frequencies can stabilize coupled oscillators even in the absence of delays whereas identical frequencies cannot [5, 6]. The analysis is based on the observation that as parameters are varied the stability of the origin may change only if an eigenvalue λ crosses the imaginary axis. In this critical situation $\lambda = i\omega$ for some real ω, and by (4)

$$(1 + i(\omega_0 - \omega) - K)^2 - K^2 \gamma^2 e^{-2i\omega\tau} = 0 \qquad (5)$$

where

$$\gamma = \gamma(\omega, \alpha) = \begin{cases} \sin(\omega\alpha)/(\omega\alpha) & \text{if } \omega\alpha \ne 0, \\ 1 & \text{if } \omega\alpha = 0. \end{cases} \qquad (6)$$

Separating (5) into real and imaginary parts and rearranging yields

$$(1 - \gamma^2)K^2 - 2K + 1 = -(\omega - \omega_0)^2 \qquad (7)$$

$$\tan(\omega\tau) = \frac{\omega - \omega_0}{1 - K}. \qquad (8)$$

This pair of equations describe a set of parametric curves on the τ-K plane in the parameter ω. For each value of ω, K is found from (7), and substitution into (8) gives the corresponding values for τ. The stability region is determined by computing the critical curves (7)–(8), and following the direction of movement of the purely imaginary eigenvalues as parameters are varied. The latter information is obtained from the quantities $\text{Re}(\partial\lambda/\partial K)$ and $\text{Re}(\partial\lambda/\partial\tau)$ calculated from (4) by implicit differentiation on the critical curves. For the parameter values obtained by this procedure, amplitude death is independently confirmed by numerical simulation of the coupled system (2)–(3).

The stability region in the K-τ parameter plane is shown in Fig. 1 for uniformly distributed delays over $\tau \pm \alpha$, with $\tau \ge \alpha$ so that the delays involved are not
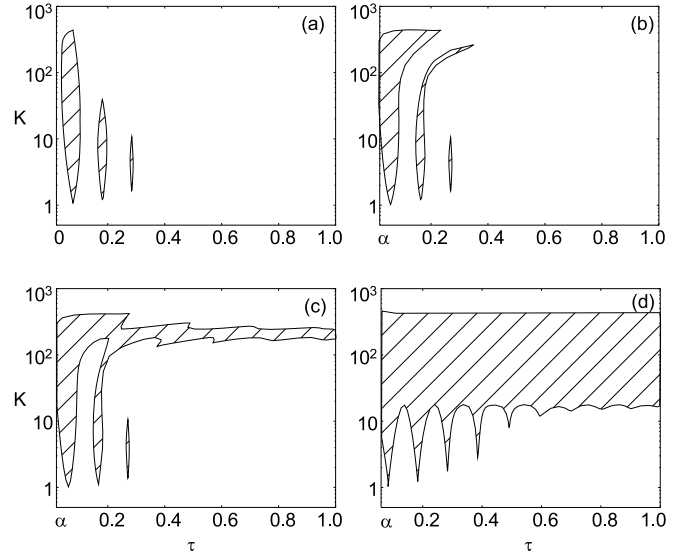

FIG. 1: Enlargement of the stability region as the spread of the delay distribution increases: (a) $\alpha = 0$, (b) $\alpha = 0.007$, (c) $\alpha = 0.008$, and (d) $\alpha = 0.02$. The value of ω_0 is 30.

negative. As $\alpha \to 0$, the uniform distribution approaches $\delta(\tau)$; hence α serves as a parameter to compare the discrete and the uniformly distributed delays having the same mean value τ. This is equivalent to quantifying the effects of the distribution by its standard deviation $\sigma = \alpha/\sqrt{3}$. When the delay is discrete the stability region consists of three disjoint and bounded sets (Fig. 1a) which deform continuously as α is increased from zero (Fig. 1b). The enclosed area increases with the variance of the distribution, and at some critical value the stability region becomes unbounded in the τ-direction (Fig. 1c). The critical value of α for this qualitative transition depends on ω_0 and it can be very small; it is slightly below 0.008 when $\omega_0 = 30$, corresponding to $\sigma = 4.62 \times 10^{-3}$. Increasing α further results in a connected and unbounded set of parameter values for amplitude death (Fig. 1d), which persists for all larger values of α. At this stage, there is a large interval of values for the coupling strength K which causes amplitude death regardless of the mean value τ. By contrast, the discrete delay can cause amplitude death only for a very limited range of delay values (Fig. 1a). Note that the ratio σ/τ of the standard deviation to the mean of the distribution can be quite small, showing that even a relatively small spread in delays may induce amplitude death.

Distributed delays further facilitate amplitude death through the parameter ω_0. The stability region in the parameter plane depends on the value of ω_0, and there exists a minimum value ω_{min} such that if $\omega_0 < \omega_{min}$ then amplitude death does not occur for any K or τ. A value of $\omega_{min} = 4.812$ has been reported for discrete delays [8]. Distributed delays can induce amplitude death even when ω_0 is smaller than this value, as Fig. 2 shows. The
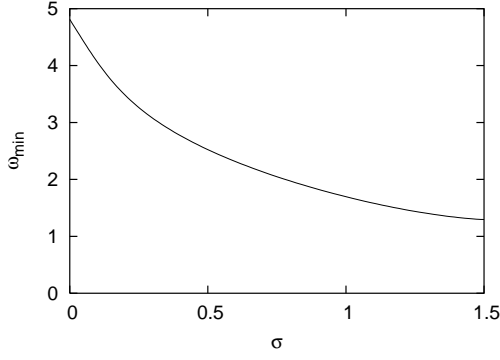
FIG. 2: Minimum value of ω_0 for which amplitude death is possible decreases as the spread of the delay distribution increases.
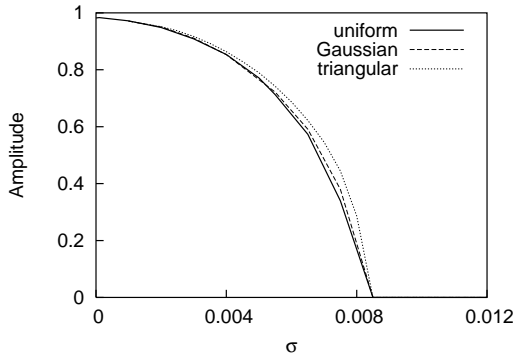

FIG. 3: Amplitudes of the limit cycles of individual oscillators versus the standard deviation of the delay distribution, for common distribution functions with the same mean value $\tau = 0.5$. Other parameters are $K = 30$ and $\omega_0 = 30$.
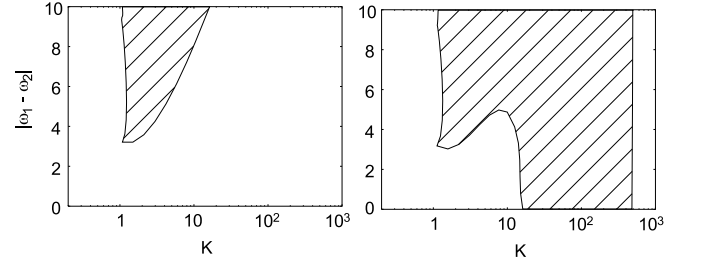

FIG. 4: Stability regions for coupled oscillators with different frequencies when the delay is (a) discrete at $\tau = 0.5$, and (b) uniformly distributed over 0.5 ± 0.02. The mean frequency of the oscillators is fixed at 30.

curve has been numerically calculated by decreasing the value of ω_0 at a fixed σ for the uniform distribution until the stability region disappears.

The local stability change caused by distributed delays is reflected in the global dynamics by the annihilation of the limit cycles of the oscillators. Fig. 3 shows the amplitudes of the oscillators, which gradually decrease with increasing standard deviation of the delay distribution. Near $\sigma = 0.0085$ the amplitude becomes zero as the limit cycle collapses to the origin and the amplitude death sets in. Numerical simulations with random initial conditions indicate that for σ beyond this value the origin is the only attractor for the coupled system. Notably, this behavior is largely independent of the particular distribution chosen for the delays, as seen in Fig. 3.

When the oscillators have different intrinsic frequencies the situation is similar, except that amplitude death occurs for even a larger parameter set, since a large frequency difference by itself is known to cause death. Fig. 4 compares the stability regions for discrete and distributed delays. As the spread of the delays increase, the stability region is enlarged and extended towards the horizontal axis, so amplitude death becomes possible also for a small (or zero) frequency difference.

The stabilizing effects of distributed delays carry over to any number of oscillators. This is illustrated in the following system of N globally coupled oscillators:

$$
\dot{Z}_j(t) = (1 + \mathrm{i}\omega_j - |Z_j(t)|^2)Z_j(t)
$$
$$
+ \frac{2K}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \left[\int_0^\infty f(\tau')Z_k(t - \tau') \, d\tau' - Z_j(t) \right] \quad (9)
$$

where $j = 1, \ldots, N$. In this model the use of distributed delays is further motivated by the fact that in a physical network the delays between units cannot be expected to be identical but can be more naturally described by a probability distribution for large N. For $N = 2$, (9) reduces to (2)–(3). The system (9) was studied in [6, 7] with no delays, and in [8] with discrete delays. For identical oscillators an analysis similar to above yields the region of amplitude death for (9). The limiting shape of this region as $N \to \infty$ is depicted in Fig. 5 for uniformly distributed delays and $\omega_j = 10$ for all j. As before, distributed delays enlarge the stability region, and there exist values of K for which amplitude death occurs regardless of the mean value τ of the delays. (The stability region is enlarged further if the frequencies ω_j are not identical, similar to the case shown in Fig. 4.) It is interesting to compare the results to those in [6, 7], where it was shown that a sufficiently large spread in the frequencies ω_j can cause amplitude death. Here a similar conclusion holds for a spread in the delays.

A detailed mathematical analysis of stability under distributed delays is too lengthy to include here; however, a brief description of the basic ideas will help clarify the role of delay distributions. Thus consider the characteristic equation (4). If $F \equiv 0$ the eigenvalues are $\lambda = 1 - K + \mathrm{i}\omega_j$, $j = 1, 2$, so the system is asymptotically stable for $K > 1$. It follows that stability can also be achieved for $|F|$ sufficiently small. Letting $\lambda = \beta + \mathrm{i}\omega$,

$$
F(\lambda) = \int_0^\infty f(\tau')e^{-\beta\tau'}[\cos\omega\tau' - \mathrm{i}\sin\omega\tau'] \, d\tau'
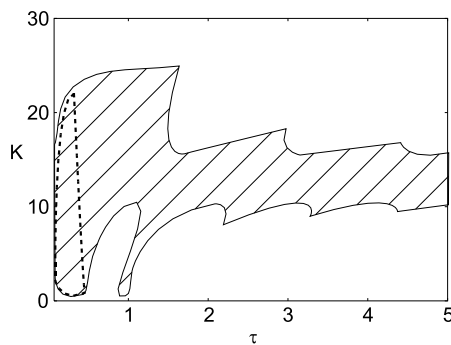$$

FIG. 5: The limiting shape of the stability region for an infinite number of globally coupled oscillators, calculated for delays uniformly distributed over $\tau \pm 0.07$. The area enclosed by the dashed line is the stability region for the discrete delay.

which can be interpreted as a (weighted) average of the quantity in brackets over an interval determined by f. If $\omega \neq 0$ the integrand is oscillatory about zero, and smaller values of $|F|$ may be obtained if the average is taken over a larger interval, i.e. if f has a large variance. It can be assured that ω is bounded away from zero provided K is not too large. In this way, an interval of values of K is obtained for which the origin is stable. A rigorous argument involves assuming $\beta \geq 0$ and obtaining a contradiction resulting from $|F|$ being small. It is also intuitively plausible that if f has a sufficiently large variance, then $|F|$ is small regardless of the mean value or the precise shape of f. This explains Figures 1d and 5, which show stability for a range of coupling strengths and arbitrary mean value of delay, as well as Fig. 3, which shows similar behavior for different distribution functions.

The importance of amplitude death has been noted by many authors in relation to various physical and biological phenomena, ranging from Belousov-Zhabotinskii reactions to cardiac arrhythmias [4, 7, 8]. For instance, the cessation of rhythmic activity in biological systems may be related to certain pathologies. Our findings suggest that in certain cases the variance of the delays, rather than their average value, could be the relevant quantity responsible for the quenching of oscillations. Delay distributions has a stabilizing effect on the interconnected system, similar to that of frequency distributions treated in previous works [6, 7]. This implies that amplitude death is a rather common and robust dynamical behavior for interacting oscillatory processes, since real-life networks inevitably involve variances in both the frequencies and the connection delays. From a practical point, the properties of distributed delays are expected to be helpful in modelling observed phenomena. For instance, a situation where amplitude death is experimentally observed but not predicted by an existing model might imply the presence of distributed delays in the physical system. Introducing a small variance in delays may in many cases lead to a better reconciliation of theory with experiments.

Finally, distributed delays give a natural way to model memory effects in interacting systems. They are thus particularly suitable in fields of neuroscience, cognition, and the general analysis of complex systems. Delays are expected to be a source of further interesting results in these highly active areas of investigation.

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